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Laser : Theory and Modern Applications

ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE (EPFL)

Homework No. 3

3.1 Fabry-Pérot Etalon The transmission T of a Fabry-Pérot Etalon is given by

$$T = \frac{I_T}{I_{\text{in}}} = \frac{1}{1 + F\sin^2(\phi/2)} \tag{1}$$

where, $\phi = 2 \cdot d \cdot \frac{\omega}{c} \cdot n$. Here d is the spacing between the mirrors, n is the refractive index of the material that fills the space between the two mirrors, and $F = \frac{4R}{(1-R)^2}$, R is the mirrors reflectivity.

1. The *Finesse* ($\mathfrak{F}$) of a Fabry-Pérot is defined as the ratio of the separation of the peaks in transmission (the free spectral range ν_{FSR}) to the spectral width of the peaks $\delta\nu_m$ (defined as the full width at half maximum, FWHM). Show that finesse is related to the mirror reflectivity by the expression below. You may assume that $F \gg 1$, since mirror reflectivities are usually high.

$$\mathfrak{F} = \frac{\pi\sqrt{R}}{1-R} \tag{2}$$

2. Calculate the free spectral range of a glass etalon (i.e. a thin piece of glass) that is 1 mm thick (the refractive index of glass is n = 1.44). Note that the reflection coefficient is given by the Fresnel law, i.e. $R = \left(\frac{n-1}{n+1}\right)^2$.

3. Suppose now that the etalon mirrors are not ideal and have finite absorption $\alpha \ll 1$. For a single mirror the reflected, absorbed and transmitted light intensities are thus related to the incident intensity I_{in} as $I_{\text{r}} = R\, I_{\text{in}}$, $I_{\text{abs}} = \alpha I_{\text{in}}$, $I_{\text{tr}} = (1 - \alpha - R)I_{\text{in}}$. Calculate how the equation (1) is modified in such case.

4. According to the the equation (1), for an ideal lossless etalon the transmission is always unity on resonance (ϕ = m 2π). What is the resonant transmission in the case when the etalon mirrors have absorption α ? What is the limitation on the minimum mirror reflectivity R necessary to observe the Fabry-Pérot resonances in such case (i.e. to have resonant transmission > 0.5) ?

3.2 Input-output relations for an optical resonator Consider again a Fabry-Pérot resonator composed of two mirrors, assuming that the mirror reflectivities r_1 and r_2 are close to 1 (corresponding transmissivities are t_1 and t_2). Here the reflectivities are given in terms of electric field amplitudes $E_r = r\,E_{\text{in}}$.

1. Let the monochromatic wave $E_{\text{in}}e^{-i\omega t}$ be incident on the mirror 1. Show that the intracavity field $E\,e^{-i\omega t}$ is related to the input field as

$$E = \frac{t_1}{1 - r_1 r_2 e^{-i\phi}} E_{\text{in}}, \tag{4}$$

where $\phi = 2\,k\,d$.

2. Assume that the laser frequency ω is close to a cavity resonance frequency ω_c, so that $\phi = m2\pi + \delta\phi$, where $m = \frac{d\,n}{\pi c}\omega_c$ is an integer and $\delta\phi = 2d\,\frac{n}{c}(\omega - \omega_c) \ll 1$. By expanding the denominator in the Eq. (4) over $\delta\phi$ show that the intracavity field amplitude has Lorenzian dependence on the detuning $\Delta = \omega - \omega_c$

$$E = \frac{\sqrt{\kappa_1}}{i\Delta + \kappa/2} \frac{E_{\text{in}}}{\sqrt{2dn/c}}, \tag{5}$$

$$\kappa = \kappa_1 + \kappa_2. \tag{6}$$

Find the constants κ_1 and κ_2 in terms of the transmissivies of the mirrors 1 and 2.

3. Show that κ_1 and κ_2 correspond to the electrical field decay rates due to the leaking through the mirrors 1 and 2 and relate the total decay rate κ to photon life time τ_p that we saw in the class.

4. Suppose now that the input field $E_{\text{in}}(t)$ is fluctuating in time. Then the Fourier transforms of the intracavity and the incident field are related as

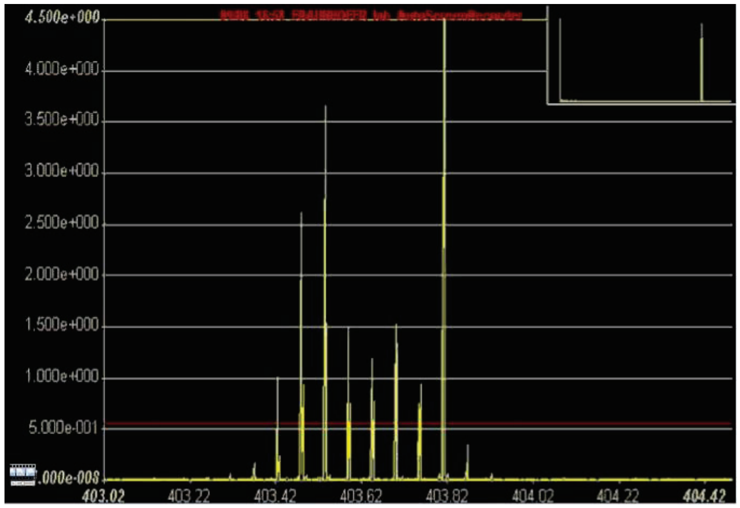
$$E[\Omega] = \frac{\sqrt{\kappa_1}}{i(\Delta - \Omega) + \kappa/2} \frac{E_{\text{in}}[\Omega]}{\sqrt{2dn/c}}. \tag{7}$$

By performing the inverse Fourier transform of the Eq. 7 find that the dynamics of the intracavity field $E(t)$ is governed by the equation

$$\frac{d}{dt}E(t) = (-i\Delta - \kappa/2)E(t) + \sqrt{\kappa_1}\frac{E_{\text{in}}(t)}{\sqrt{2dn/c}}. \tag{8}$$

The equation (8) is a convenient basis for consideration of the interaction between the intracavity field and nonlinear medium.

3.3 Multimode laser diodes : cavity length and Fabry-Pérot analyzer A blue-violet laser diode (Blu-ray laser diode) emits multiple longitudinal modes. A typical spectrum is shown in the graph below using a Fabry-Pérot. The horizontal axis is in nanometers.



1. Find the laser cavity length of the semi-conductor laser that gives the mode spacing shown in the graph (use refractive index $n = 2.6$ for the semi-conductor InGaN/GaN).
2. Design a Fabry-Pérot etalon (i.e find the mirror reflectivities and mirror spacing) such that the spectral range of interest is 1 nm near 405 nm. Take 10% of the laser cavity mode spacing as the resolution of the etalon. The etalon operates in air ($n = 1$).
3. Estimate the coherence length of the laser diode using the graph.

3.4 In a four level system, the population rate is given by

$$\begin{aligned}
\frac{dN_0}{dt} &= -PN_0 + \Gamma_{10} \cdot N_1 \\
\frac{dN_1}{dt} &= -\Gamma_{10} \cdot N_1 + \Gamma_{21} N_2 + \sigma(\nu)(N_2 - N_1)\phi_\nu \\
\frac{dN_2}{dt} &= PN_0 - \Gamma_{21} N_2 - \sigma(\nu)(N_2 - N_1)\phi_\nu
\end{aligned}$$

1. Derive the steady-state population inversion shown below. At steady state $\frac{d\overline{N}_0}{t} = \frac{d\overline{N}_1}{t} = \frac{d\overline{N}_2}{t} = 0$ and neglecting ϕ_ν

$$\overline{N}_2 - \overline{N}_1 = \frac{P(\Gamma_{10} - \Gamma_{21})N_T}{\Gamma_{10} \cdot \Gamma_{21} + \Gamma_{10} \cdot P + \Gamma_{21} \cdot P}$$

2. The steady state population inversion of a three-level system is given by :

$$\overline{N}_2 - \overline{N}_1 = \frac{P - \Gamma_{21}}{P + \Gamma_{21}} \cdot N_T$$

Derive the ratio of pumping power between a four and a three level system necessary to achieve the population inversion ΔN_T. $N_T = N_1 + N_2 + N_3$ is the total number of atoms involved.

$$\frac{(P)_{\text{4-level laser}}}{(P)_{\text{3-level laser}}} = \frac{\Delta N_T}{\Delta N_T + N_T}$$

What do you notice ?

3.5 Threshold of Nd :YAG, Ruby and He-Ne laser We wish to compare the population inversion threshold of three laser systems, namely Nd :YAG (Neodymium ions), Ruby (chromium ions) and Helium-Neon (Neon) lasers. The first two are solid-state lasers while the third is a gas laser. A ruby laser is approximately a three level system whereas Nd :YAG and He-Ne are four level systems.

1. Calculate the population inversion threshold for each laser. What do you observe between solid state and gas lasers ?
2. For a three level system the required power per unit volume of laser material to reach threshold is approximately $(P/V) = 1/2 \cdot h \cdot \nu_{31} \cdot N_T \cdot \Gamma_{21}$ where ν_{31} is the frequency of the pump light ($\lambda = 505nm$ for Ruby), N_T the density of active laser ions and Γ_{21} the non radiative decay rate. The concentration of Chromium in Ruby is 0.05% weight percent.

TABLE 1 – Laser medium characteristics

	Absorption/ Stimulated Cross section $[cm^2]$	Spontaneous emission rate A [1/s]	Linewidth [MHz]	Non radiative decay rate Γ_{21} [1/s]	Transition wavelength [nm]
Nd :YAG	$3 \cdot 10^{-19}$	-	120,000	4,400	1064
Ruby	$2.7 \cdot 10^{-20}$	230	170,000	500	694.3
He-Ne	$1.4 \cdot 10^{-17}$	$1.4 \cdot 10^{6}$	1,500	-	632.8

TABLE 2 – Laser resonator characteristics

	Scattering loss per round trip [%]	Resonator length [cm]	Laser rod radius [mm]	Mirror reflectivities R1 [1/s]	Mirror reflectivities R2 [nm]
Nd :YAG	3	10	2	1	0.96
Ruby	3	5	2	1	0.96
He-Ne	0.2	30	-	0.998	0.98

For a four level system the required power per unit volume of laser material to reach threshold is approximately $(P/V) = 1/2 \cdot h \cdot \nu_{30} \cdot \Delta N_T \cdot \Gamma_{21}$ where ν_{30} is the frequency of the pump light ($\lambda = 808nm$ for YAG), ΔN_T the inversion population at threshold and Γ_{21} the non radiative decay rate.

3.6 Cavity frequency pulling effect The condition for laser oscillation frequency is modified from the bare cavity frequencies, due to the fact that an atomic gain medium has dispersion due to Kramers Kroenig relation encountered earlier, i.e. $(\nu) - 1 = -\lambda_{21}/4\pi \frac{\nu_{21}-\nu}{\delta\nu_{21}} \cdot g(\nu)$. Show that due to this effect the actual laser frequency is being pulled towards the atomic center frequency (ν_{21}) - an effect called cavity mode pulling. Do so by deriving the frequency for laser oscillation in the presence of an inverted, homogeneously broadened atomic gain medium, and use gain clamping condition above threshold. Show that the laser oscillation occurs (for $\delta\nu_{21} \gg \delta\nu_v$) approximately at $\nu = \nu_m + (\nu_{22} - \nu_m)\frac{\delta\nu_c}{\delta\nu_{21}}$, where the cavity decay rate is $\delta\nu_c$.